DEFINITIONS OF $h-$LOGARITMIC, $h-$GEOMETRIC AND $h-$MULTI CONVEX FUNCTIONS AND SOME INEQUALITIES RELATED TO THEM

M.EMİN ÖZDEMİR★, MEVLÜT TUNÇ■, AND MUSTAFA GÜRBÜZ▲

ABSTRACT. In this paper, we put forward some new definitions and integral inequalities by using fairly elementary analysis.

1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality [1].

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f: I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

The following definitions is well known in the literature.

Definition 1. *A function $f: I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, where I is a convex set, is said to be convex on I if inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.2}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The concept of h-convexity was introduced by Varošanec [12] and was generalized by Házy [21].

Definition 2. [12] *Let $h: J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f: I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y). \tag{1.3}$$

If inequality (1.3) is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$. Obviously, if $h(t) = t$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$.

Definition 3. [4] *A function $h: J \to \mathbb{R}$ is said to be a superadditive function if*

$$h(x+y) \geq h(x) + h(y) \tag{1.4}$$

for all $x, y \in J$.

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▲Corresponding Author.

Recently, In [2], the concept of geometrically and s-geometrically convex functions was introduced as follows.

Definition 4. [2] *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a geometrically convex function if*

$$f\left(x^t y^{1-t}\right) \leq [f(x)]^t [f(y)]^{1-t} \tag{1.5}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 5. [2] *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an s-geometrically convex function if*

$$f\left(x^t y^{1-t}\right) \leq [f(x)]^{t^s} [f(y)]^{(1-t)^s} \tag{1.6}$$

for some $s \in (0,1]$, $x, y \in I$ and $t \in [0,1]$.

If $s = 1$, the s-geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.

In [3], Tunç and Akdemir introduced the class of s-logarithmically convex functions in the first and second sense as the following:

Definition 6. *A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be s-logarithmically convex in the first sense if*

$$f(\alpha x + \beta y) \leq [f(x)]^{\alpha^s} [f(y)]^{\beta^s} \tag{1.7}$$

for some $s \in (0,1]$, where $x, y \in I$ and $\alpha^s + \beta^s = 1$.

Definition 7. *A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be s-logarithmically convex in the second sense if*

$$f(tx + (1-t)y) \leq [f(x)]^{t^s} [f(y)]^{(1-t)^s} \tag{1.8}$$

for some $s \in (0,1]$, where $x, y \in I$ and $t \in [0,1]$.

Clearly, when taking $s = 1$ in Definition 6 or Definition 7, then f becomes the standard logarithmically convex function on I.

2. Results for h-log-Convex Functions

Definition 8. *A positive function f is called h-logarithmically convex on a real interval $I = [a, b]$, if for all $x, y \in I$ and $t \in [0,1]$,*

$$f(tx + (1-t)y) \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)} \tag{2.1}$$

where $h(t)$ is a nonnegative function on J, with $h : J \subseteq \mathbb{R} \to \mathbb{R}$.

If f is a positive h-logarithmically concave function, then inequality is reversed. On the other hand, a function f is h-logarithmically convex on I if f is positive and $\log f$ is h-convex on I.

Proof. Let's rewrite $g = \log f(x)$. Since g is $h-$convex function, for $x, y \in I$ and $t \in [0,1]$ we get

$$\begin{aligned} g(tx + (1-t)y) &\leq h(t) g(x) + h(1-t) g(y) \\ \log f(tx + (1-t)y) &\leq h(t) \log f(x) + h(1-t) \log f(y) \\ &= \log f(x)^{h(t)} + \log f(y)^{h(1-t)} \end{aligned}$$

So we have

$$\begin{aligned} f(tx + (1-t)y) &\leq e^{\log f(x)^{h(t)}} . e^{\log f(y)^{h(1-t)}} \\ &= [f(x)]^{h(t)} [f(y)]^{h(1-t)} . \end{aligned}$$

□

Remark 1. *If we take $h(t) = t$ in Definition 8, h-logarithmically convex (concave) become ordinary* log-*convex (concave) function, and if we take $h(t) = t^s$ in Definition 8, h-logarithmically convex (concave) become s*-log-*convex (concave) function in the second sense.*

Proposition 1. *Let f be an h*-log-*convex function. If the function h satisfies the condition*

$$h(t) + h(1-t) = 1$$

for all $t \in [0,1]$, then f is also h-convex function.

Proof. As we choose f is h-log-convex function we can write

$$f(tx + (1-t)y) \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)}.$$

From a simple inequality

$$x^{\alpha} y^{1-\alpha} \leq \alpha x + (1-\alpha) y$$

for $x, y > 0$ and by using the condition $h(t) + h(1-t) = 1$ we have

$$\begin{aligned} f(tx + (1-t)y) &\leq [f(x)]^{h(t)} [f(y)]^{h(1-t)} \\ &\leq h(t) f(x) + h(1-t) f(y) \end{aligned}$$

which shows that f is h-convex function. □

Theorem 1. *Let f be an h*-log-*convex function. If f is monotonically increasing or decreasing and h is superadditive function on $[0,1]$, we have*

$$\frac{1}{b-a} \int_a^b f(x) f(a+b-x)\, dx \leq [f(a) f(b)]^{h(1)} \tag{2.2}$$

and

$$\frac{1}{(b-a)^2} \left(\int_a^b f(x)\, dx \right)^2 \leq [f(a) f(b)]^{h(1)}. \tag{2.3}$$

Proof. Since f is an h-log-convex function, for $a, b \in I$, $t \in [0,1]$ we have

$$f(ta + (1-t)b) \leq [f(a)]^{h(t)} [f(b)]^{h(1-t)}$$

and

$$f(tb + (1-t)a) \leq [f(b)]^{h(t)} [f(a)]^{h(1-t)}.$$

If we multiply both sides we have

$$\begin{aligned} f(ta + (1-t)b) f(tb + (1-t)a) &\leq [f(a) f(b)]^{h(t)} [f(a) f(b)]^{h(1-t)} \\ &= [f(a) f(b)]^{h(t)+h(1-t)}. \end{aligned}$$

As we choose h is superadditive function we get

$$f(ta + (1-t)b) f(tb + (1-t)a) \leq [f(a) f(b)]^{h(1)}.$$

By integrating the last inequality over t from 0 to 1 we get

$$\frac{1}{b-a} \int_a^b f(x) f(a+b-x)\, dx \leq [f(a) f(b)]^{h(1)}.$$

So the proof of (2.2) is completed.

On the other hand is we use Chebyshev inequality on (2.2) we have

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) f(a+b-x)\, dx &\geq \frac{1}{(b-a)^2} \int_a^b f(x)\, dx \int_a^b f(a+b-x)\, dx \\ &= \frac{1}{(b-a)^2} \left(\int_a^b f(x)\, dx \right)^2. \end{aligned}$$

So we have

$$\frac{1}{(b-a)^2}\left(\int_a^b f(x)\,dx\right)^2 \leq [f(a)\,f(b)]^{h(1)}.$$

Then proof of (2.3) is completed. □

Corollary 1. *If we choose $h(t)=\frac{1}{t}$ at Theorem 1 as a superadditive function on $[0,1]$ we have*

$$\frac{1}{b-a}\int_a^b f(x)\,f(a+b-x)\,dx \leq f(a)\,f(b)$$

and

$$\frac{1}{(b-a)^2}\left(\int_a^b f(x)\,dx\right)^2 \leq f(a)\,f(b).$$

Theorem 2. *Let f and g are h-log-convex functions on I and let h is symmetric about $\frac{1}{2}$. For $a,b\in I$ and $t\in[0,1]$ we have*

$$\frac{1}{b-a}\int_a^b (fg)(x)\,dx \leq \int_0^1 [(fg)(a)\,(fg)(b)]^{h(t)}\,dt. \tag{2.4}$$

Proof. As we choose f and g are h-log-convex functions on I we have

$$\begin{aligned} f(ta+(1-t)b) &\leq [f(a)]^{h(t)}\,[f(b)]^{h(1-t)} \\ g(ta+(1-t)b) &\leq [g(a)]^{h(t)}\,[g(b)]^{h(1-t)}. \end{aligned}$$

If we multiply both sides we get

$$f(ta+(1-t)b)\,g(ta+(1-t)b) \leq [f(a)\,g(a)]^{h(t)}\,[f(b)\,g(b)]^{h(1-t)}.$$

By integrating the inequality from 0 to 1 over t, and change the variable $x=ta+(1-t)b$ we have

$$\frac{1}{b-a}\int_a^b (fg)(x)\,dx \leq \int_0^1 [f(a)\,g(a)]^{h(t)}\,[f(b)\,g(b)]^{h(1-t)}\,dt.$$

Since h is symmetric about $\frac{1}{2}$ we have $h(t)=h(1-t)$. So we have

$$\frac{1}{b-a}\int_a^b (fg)(x)\,dx \leq \int_0^1 [(fg)(a)\,(fg)(b)]^{h(t)}\,dt.$$

□

Theorem 3. *Let f and g are h-log-convex functions. For $\alpha,\beta>0$ and $\alpha+\beta=1$ we have*

$$\frac{1}{b-a}\int_a^b (fg)(x)\,dx \leq \int_0^1 \left[\alpha\left\{[f(a)]^{h(t)}\,[f(b)]^{h(1-t)}\right\}^{\frac{1}{\alpha}} + \beta\left\{[g(a)]^{h(t)}\,[g(b)]^{h(1-t)}\right\}^{\frac{1}{\beta}}\right]dt \tag{2.5}$$

and

$$\frac{1}{b-a}\int_a^b (fg)(x)\,dx \leq \int_0^1 \left\{\alpha\,[f(a)\,g(a)]^{\frac{h(t)}{\alpha}} + \beta\,[f(b)\,g(b)]^{\frac{h(1-t)}{\beta}}\right\}dt. \tag{2.6}$$

Proof. Since f and g are h-log-convex functions we have

$$\begin{aligned} f(ta+(1-t)b) &\leq [f(a)]^{h(t)}\,[f(b)]^{h(1-t)} \qquad (2.7)\\ g(ta+(1-t)b) &\leq [g(a)]^{h(t)}\,[g(b)]^{h(1-t)}. \end{aligned}$$

If we multiply both sides and use the fact that $cd\leq \alpha c^{\frac{1}{\alpha}}+\beta d^{\frac{1}{\beta}}$ (for $\alpha,\beta>0$, $\alpha+\beta=1$) we get

$$(fg)(ta+(1-t)b) \leq \alpha\left\{[f(a)]^{h(t)}\,[f(b)]^{h(1-t)}\right\}^{\frac{1}{\alpha}} + \beta\left\{[g(a)]^{h(t)}\,[g(b)]^{h(1-t)}\right\}^{\frac{1}{\beta}}.$$

By integrating the above inequality, we get the proof of (2.5).

On the other hand after multiplying both sides of (2.7) we can write

$$(fg)(ta+(1-t)b) \leq \alpha [f(a)g(a)]^{\frac{h(t)}{\alpha}} + \beta [f(b)g(b)]^{\frac{h(1-t)}{\beta}}.$$

Then, by integrating the last inequality we get the proof of 2.6. $\square$

Theorem 4. *Let f be an h-log-convex function on $[a,b]$. For $\alpha,\beta>0$, $\alpha+\beta=1$ we have*

$$f\left(\frac{a+b}{2}\right) \leq \alpha \frac{1}{b-a}\int_a^b f(x)^{\frac{h\left(\frac{1}{2}\right)}{\alpha}} dx + \beta \frac{1}{b-a}\int_a^b f(x)^{\frac{h\left(\frac{1}{2}\right)}{\beta}} dx.$$

Proof. If we choose $t=\frac{1}{2}$ on Definition 8 we have

$$f\left(\frac{x+y}{2}\right) \leq [f(x)f(y)]^{h\left(\frac{1}{2}\right)}$$

If we change the variable $x=ta+(1-t)b$ and $y=(1-t)a+tb$ we get

$$f\left(\frac{a+b}{2}\right) \leq f(ta+(1-t)b)^{h\left(\frac{1}{2}\right)} f((1-t)a+tb)^{h\left(\frac{1}{2}\right)}$$

If we use the inequality $cd \leq \alpha c^{\frac{1}{\alpha}} + \beta d^{\frac{1}{\beta}}$ (for $\alpha,\beta>0$, $\alpha+\beta=1$) we get

$$f\left(\frac{a+b}{2}\right) \leq \alpha f(ta+(1-t)b)^{\frac{h\left(\frac{1}{2}\right)}{\alpha}} + \beta f((1-t)a+tb)^{\frac{h\left(\frac{1}{2}\right)}{\beta}}.$$

By integrating the last inequality over t on $[0,1]$ we have

$$f\left(\frac{a+b}{2}\right) \leq \alpha \int_0^1 f(ta+(1-t)b)^{\frac{h\left(\frac{1}{2}\right)}{\alpha}} dt + \beta \int_0^1 f((1-t)a+tb)^{\frac{h\left(\frac{1}{2}\right)}{\beta}} dt.$$

By rewriting the inequality by using suitable variable changings we get the desired result. $\square$

3. Results for h-geometrically Convex Functions

Definition 9. *A positive function f is called h-geometrically convex on a real interval $I=[a,b]$, if for all $x,y\in I$ and $t\in[0,1]$,*

$$f\left(x^t y^{(1-t)}\right) \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)} \tag{3.1}$$

where $h(t)$ is a nonnegative function on J, with $h: J\subseteq \mathbb{R}\to\mathbb{R}$.

If f is a positive h-geometrically concave function, then inequality is reversed.

Remark 2. *It is clear that when $h(t)=t$ in Definition 9, h-geometrically convex (concave) become ordinary geometrically convex (concave) function, and if we take $h(t)=t^s$ in Definition 9, h-geometrically convex (concave) become s-geometrically convex (concave) function.*

Remark 3. *As we can write*

$$x^t y^{(1-t)} \leq tx+(1-t)y$$

for $t\in[0,1]$ and $x,y>0$, we get all Theorems and Corollaries given at Section 2 for decreasing $h-$geometrically convex functions.

Theorem 5. *Let f be an h-geometrically convex function on I. For every $x,y\in I$ with $x<y$ we get*

$$\frac{1}{\ln y - \ln x}\int_x^y f(\gamma) f\left(\frac{xy}{\gamma}\right)\frac{d\gamma}{\gamma} \leq \int_0^1 [f(x)f(y)]^{h(t)+h(1-t)} dt.$$

Proof. Since we choose f is an h-geometrically convex function on I, we can write

$$\begin{aligned} f\left(x^t y^{1-t}\right) &\leq [f(x)]^{h(t)} [f(y)]^{h(1-t)} \\ f\left(x^{1-t} y^t\right) &\leq [f(x)]^{h(1-t)} [f(y)]^{h(t)}. \end{aligned}$$

If we multiply both sides of inequalities we get

$$f\left(x^t y^{1-t}\right) f\left(x^{1-t} y^t\right) \leq [f(x) f(y)]^{h(t)+h(1-t)}$$

By integrating both sides respect to t over $[0,1]$ we have

$$\int_0^1 f\left(x^t y^{1-t}\right) f\left(x^{1-t} y^t\right) dt \leq \int_0^1 [f(x) f(y)]^{h(t)+h(1-t)} dt$$

If we change the variable $\gamma = x^t y^{1-t}$, we get the desired result. □

Theorem 6. *Let f and g are h-geometricially convex functions on I. For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ we get*

$$\begin{aligned} \int_0^1 f\left(x^t y^{1-t}\right) g\left(x^{1-t} y^t\right) dt &\leq \left(\int_0^1 f(x)^{p^2 h(t)} dt\right)^{\frac{1}{p^2}} \left(\int_0^1 g(y)^{pqh(t)} dt\right)^{\frac{1}{pq}} \\ &\quad \times \left(\int_0^1 f(y)^{pqh(1-t)} dt\right)^{\frac{1}{pq}} \left(\int_0^1 g(x)^{q^2 h(1-t)} dt\right)^{\frac{1}{q^2}} \end{aligned}$$

for every $x, y \in I$ with $x < y$ and $t \in [0,1]$.

Proof. As we choose f and g are h-geometricially convex functions on I we can write

$$\begin{aligned} f\left(x^t y^{1-t}\right) &\leq [f(x)]^{h(t)} [f(y)]^{h(1-t)} \\ g\left(x^{1-t} y^t\right) &\leq [g(x)]^{h(1-t)} [g(y)]^{h(t)}. \end{aligned}$$

By multiplying both sides and integrate respect to t over $[0,1]$ we have

$$\int_0^1 f\left(x^t y^{1-t}\right) g\left(x^{1-t} y^t\right) dt \leq \int_0^1 [f(x) g(y)]^{h(t)} [f(y) g(x)]^{h(1-t)} dt.$$

If we apply Hölder's inequality for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ we get

$$\int_0^1 f\left(x^t y^{1-t}\right) g\left(x^{1-t} y^t\right) dt \leq \left(\int_0^1 [f(x) g(y)]^{ph(t)} dt\right)^{\frac{1}{p}} \left(\int_0^1 [f(y) g(x)]^{qh(1-t)} dt\right)^{\frac{1}{q}}.$$

Then by applying Hölder's inequality again, we get

$$\begin{aligned} \int_0^1 f\left(x^t y^{1-t}\right) g\left(x^{1-t} y^t\right) dt &\leq \left[\left(\int_0^1 f(x)^{p^2 h(t)} dt\right)^{\frac{1}{p}} \left(\int_0^1 g(y)^{pqh(t)} dt\right)^{\frac{1}{q}}\right]^{\frac{1}{p}} \\ &\quad \times \left[\left(\int_0^1 f(y)^{pqh(1-t)} dt\right)^{\frac{1}{p}} \left(\int_0^1 g(x)^{q^2 h(1-t)} dt\right)^{\frac{1}{q}}\right]^{\frac{1}{q}}. \end{aligned}$$

By rearranging the inequality, we get the desired rusult. □

4. h-multi Convex Functions

Finally, we can introduce the following definition.

Definition 10. *A positive function f is called h-multi convex on a real interval $I = [a,b]$, if for all $x, y \in I$ and $t, \lambda \in [0,1]$,*

$$\lambda f\left(x^t y^{(1-t)}\right) + (1-\lambda) f(tx + (1-t)y) \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)} \tag{4.1}$$

where $h(t)$ is a nonnegative function on J, with $h : J \subseteq \mathbb{R} \to \mathbb{R}$.

If f is a positive h-multi concave function, then inequality is reversed.

Remark 4. *It is clear that when $\lambda = 0$ in Definition 10, h-multi convex (concave) become h-logarithmically convex (concave) function, and if we take $\lambda = 1$ in Definition 10, h-multi convex (concave) become s-geometrically convex (concave) function.*

Theorem 7. *Let f be an $h-$multi convex function on I. Then we get*

$$\frac{1}{2}\left[\frac{1}{\ln y - \ln x}\int_x^y \frac{f(\gamma)}{\gamma}d\gamma + \frac{1}{y-x}\int_x^y f(\gamma)\,d\gamma\right] \leq \int_0^1 [f(x)]^{h(t)}[f(y)]^{h(1-t)}\,dt$$

for all $x, y \in I$ and $t, \lambda \in [0,1]$.

Proof. From Definiton 10 we have

$$\lambda f\left(x^t y^{(1-t)}\right) + (1-\lambda) f(tx + (1-t)y) \leq [f(x)]^{h(t)}[f(y)]^{h(1-t)}.$$

If we integrate the inequality respect to λ over $[0,1]$ we have

$$\int_0^1 \left(\lambda f\left(x^t y^{(1-t)}\right) + (1-\lambda) f(tx + (1-t)y)\right) d\lambda \leq \int_0^1 [f(x)]^{h(t)}[f(y)]^{h(1-t)}\,d\lambda.$$

So we have

$$\frac{f\left(x^t y^{(1-t)}\right) + f(tx + (1-t)y)}{2} \leq [f(x)]^{h(t)}[f(y)]^{h(1-t)}.$$

Then by integrating the inequality respect to t over $[0,1]$ we get the desired result. □

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★ATATÜRK UNIVERSITY, K. K. EDUCATION FACULTY, DEPARTMENT OF MATHEMATICS, 25240, CAMPUS, ERZURUM, TURKEY
E-mail address: emos@atauni.edu.tr

■UNIVERSITY OF KİLİS 7 ARALIK, FACULTY OF ARTS AND SCIENCES, DEPARTMENT OF MATHEMATICS, 79000, KİLİS, TURKEY
E-mail address: mevluttunc@kilis.edu.tr

▲ UNIVERSITY OF AĞRI İBRAHİM ÇEÇEN, EDUCATION FACULTY, DEPARTMENT OF MATHEMATICS, 04100, AĞRI, TURKEY
E-mail address: mgurbuz@agri.edu.tr